

Today (9/28/01)

- Get started on Project 2! You should be able to do parts 1-3 right now.
- Today
 - Root Locus
 - Ref: 5.1-5.3

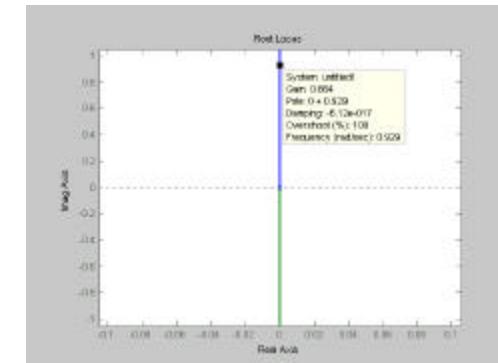
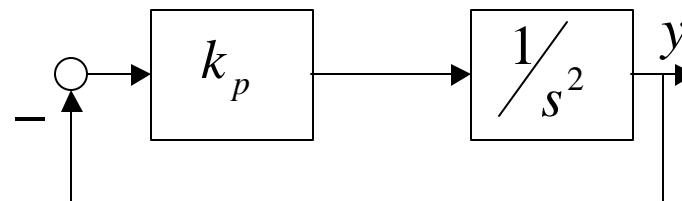
Root Locus

- Plot the closed loop poles as a function of a *single* gain.

$$s^2 + k_p = 0$$

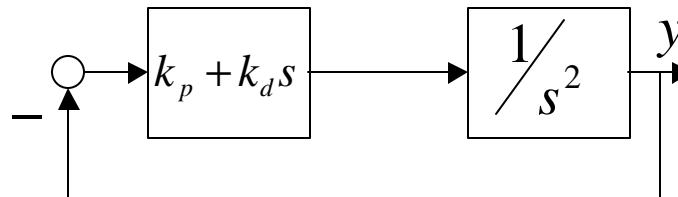
$$1 + L(s) = 1 + \frac{k_p}{s^2}$$

$L(s)$ = loop gain

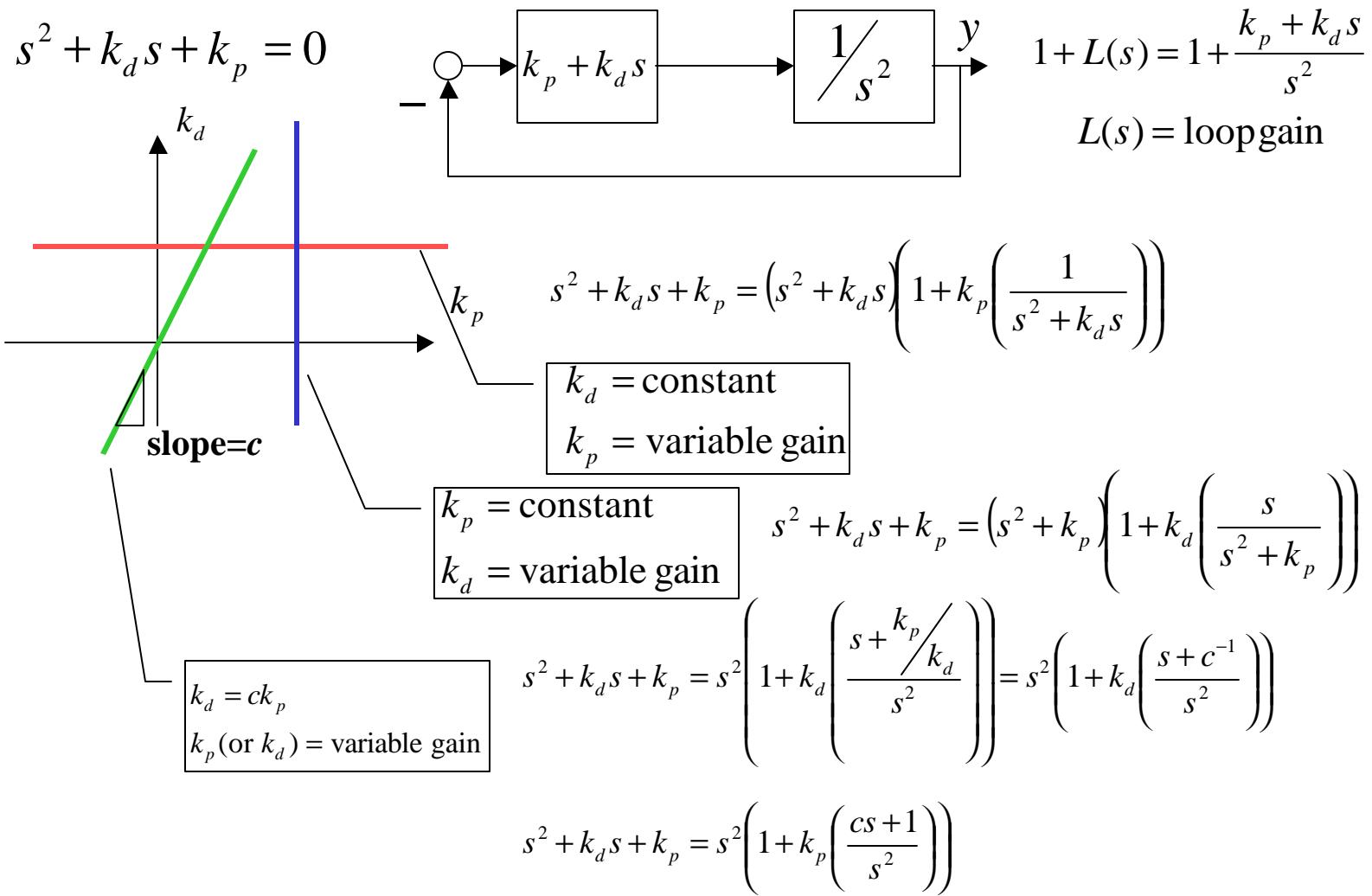


- What if there are more than one gains?

$$s^2 + k_d s + k_p = 0$$



Multiple Gains



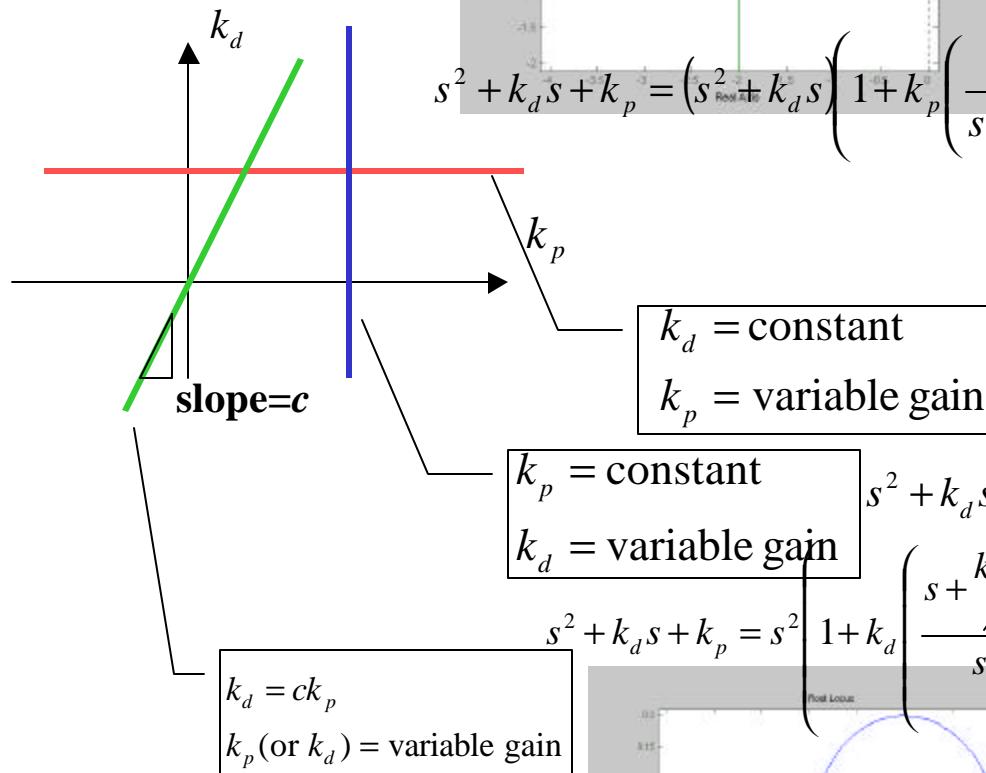
Multiple Gains

- Choose one variable gain and fix relationship between all other gains and the variable gain.
- Write the characteristic equation (i.e., numerator of $1+L(s)$) as $M(s)(1+kF(s))$ where k = variable gain, $M(s)$, $F(s)$ are known transfer functions.

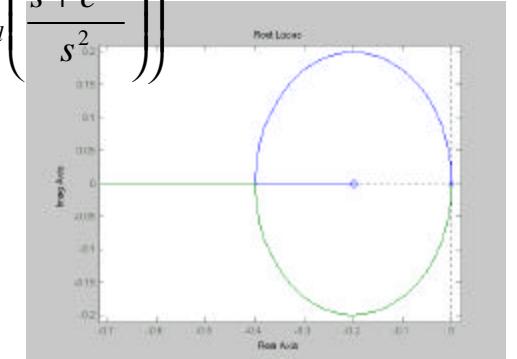
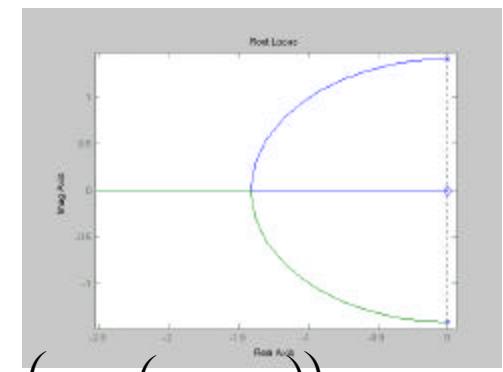
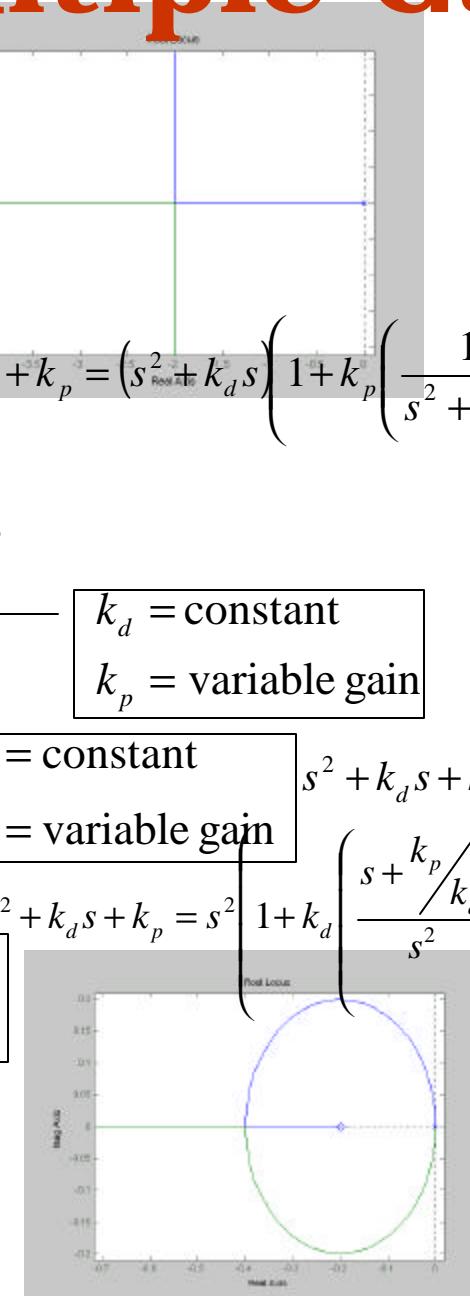
From now on, we will only consider the standard form: $1+k F(s)$ where $F(s)$ is given.

Multiple Gains

$$s^2 + k_d s + k_p = 0$$



$$s^2 + k_d s + k_p = s^2 \left(1 + k_p \left(\frac{cs + 1}{s^2} \right) \right)$$



Rules for Root Locus

Write $F(s) = b(s)/a(s)$ (convention: leading coefficient of $b(s)$ is always positive).

$$1 + k \frac{-s+1}{s^2+4} \rightarrow 1 + \underbrace{(-k)}_{\text{new } k} \frac{s-1}{s^2+4}$$

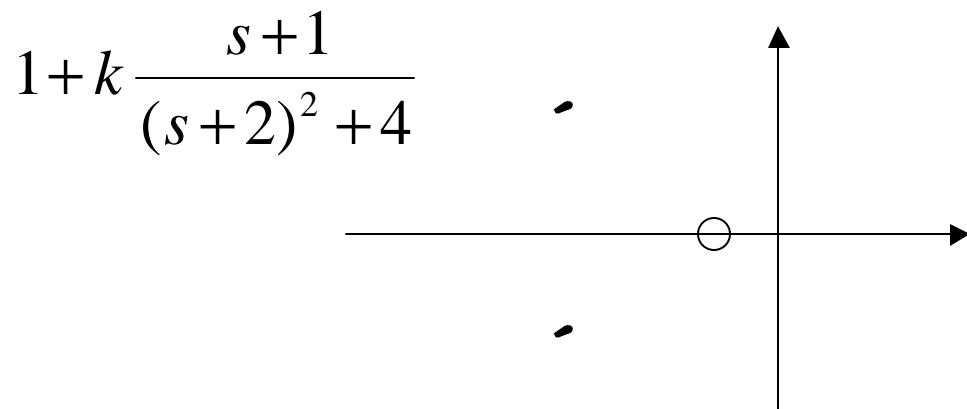
$$1 + kF(s) = 1 + k \frac{b(s)}{a(s)} = \frac{a(s) + kb(s)}{a(s)}$$

characteristic polynomial : $a(s) + kb(s)$

when $k = 0$, closed loop poles = open loop poles

Steps for Drawing Root Locus

Step 1: Draw on the complex plane open loop poles and zeros (poles/zeros of $F(s)$, roots of $a(s)$ and $b(s)$). use **x** for poles and **o** for zeros.



Property of Root Locus

$$1 + kF(s) = 0 \Leftrightarrow \underbrace{F(s)}_{\text{complex}} = -\frac{1}{\underbrace{k}_{\text{real}}} \quad \therefore \angle F(s) = \begin{cases} 180^\circ \pm l 360^\circ & \text{if } k > 0 \\ 0^\circ \pm l 360^\circ & \text{if } k < 0 \end{cases}$$

Terminology :

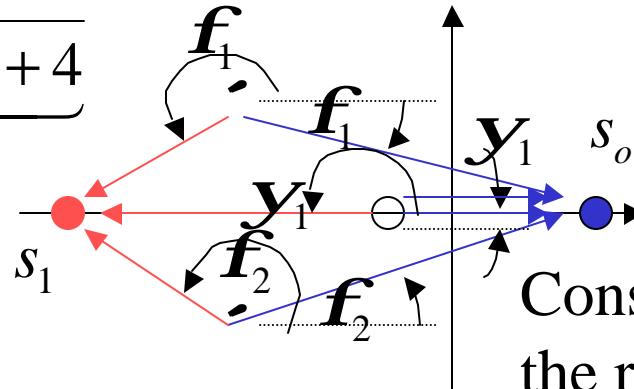
$k > 0 : 180^\circ$ locus

$k < 0 : 0^\circ$ locus

This means: In order for a complex number s to be on the 180° locus, phase of $F(s) = 180^\circ \pm l 360^\circ$; s is on the 0° locus if phase of $F(s) = l 360^\circ$.

Property of Root Locus

$$1 + k \underbrace{\frac{s+1}{(s+2)^2 + 4}}_{F(s)}$$



Consider any s_o on the real axis.

$$\angle F(s_o) = \mathbf{y}_1 - \mathbf{f}_1 - \mathbf{f}_2 = 0^\circ$$

Therefore, s_o is on the 0° locus.

When is s_o on the 180° locus?

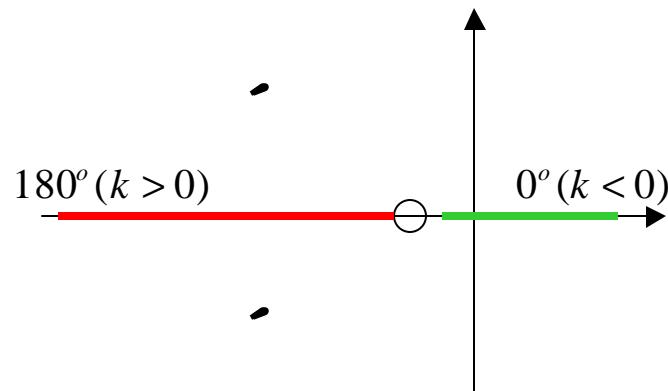
$$\angle F(s_o) = \mathbf{y}_1 - \mathbf{f}_1 - \mathbf{f}_2 = 180^\circ$$

Therefore, s_1 is on the 180° locus.

When is s_o on the 0° locus?

Steps for Drawing Root Locus

Step 2: Consider any s on the real axis. If there is an *odd* number of *real* poles and zeros to the right of s , then s is on the 180° root locus. Otherwise, s is on the 0° root locus.



Steps for Drawing Root Locus

Step 3: Asymptotic root loci. Consider $k \rightarrow \infty$ and $k \rightarrow -\infty$.

$$1 + kF(s) = 1 + k \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

As $k \rightarrow \pm \infty$, $1 + kF(s) = 0 \Rightarrow F(s) = -1/k \Rightarrow F(s) \rightarrow 0$

Therefore, m closed loop poles $\rightarrow m$ zeros of $F(s)$
 $n-m$ closed loop poles $\rightarrow \infty$. How do these poles approach ∞ ?

Steps for Drawing Root Locus

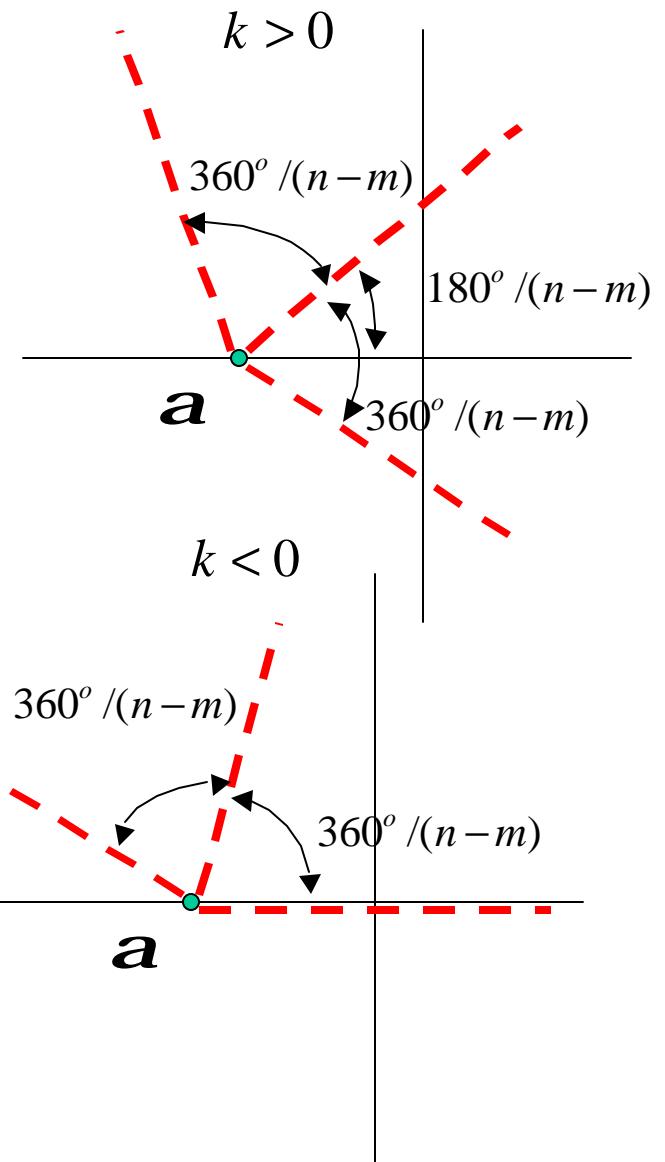
For large value of $s : F(s) \approx \frac{1}{(s-a)^{n-m}}$

$$a = \frac{\sum p_i - \sum z_i}{n-m}$$

Recall: $\angle F(s) = \begin{cases} 180^\circ \pm l360^\circ & \text{if } k > 0 \\ 0^\circ \pm l360^\circ & \text{if } k < 0 \end{cases}$

$$k > 0 : \angle(s-a) = \frac{180^\circ + l360^\circ}{n-m}, l = 1, \dots, n-m$$

$$k < 0 : \angle(s-a) = \frac{-l360^\circ}{n-m}, l = 1, \dots, n-m$$



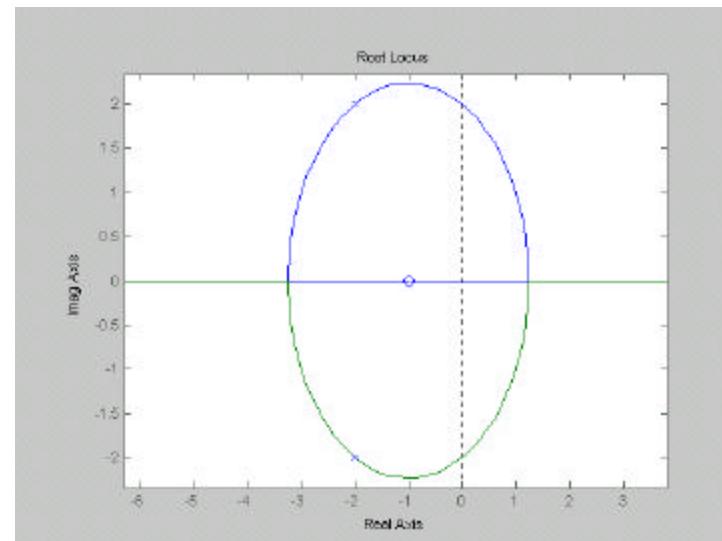
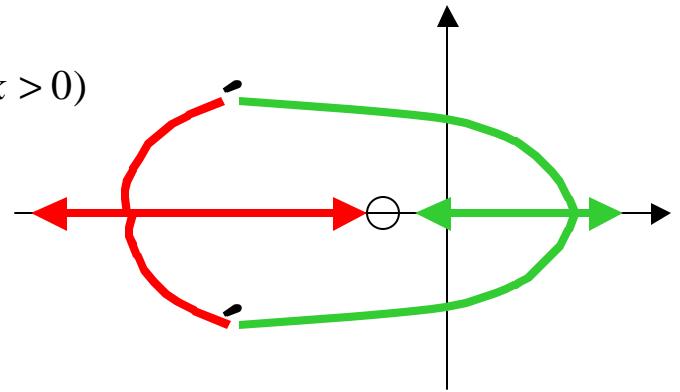
Example

$$1 + k \frac{s+1}{(s+2)^2 + 4}$$

$$m=1, n=2, n-m=1$$

$$a=-3$$

$180^\circ (k > 0)$ $0^\circ (k < 0)$



Example

$$1 + k \frac{s+1}{s^2(s+12)}$$

$$m=1, n=3, n-m=2$$

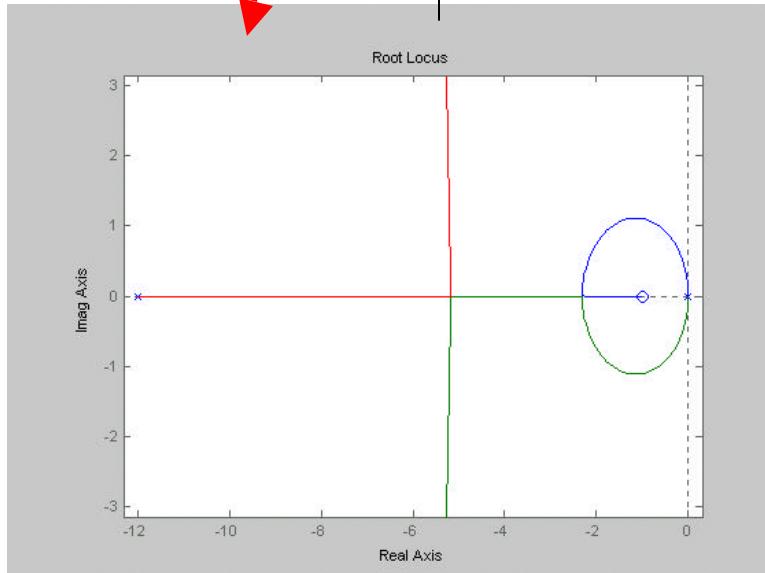
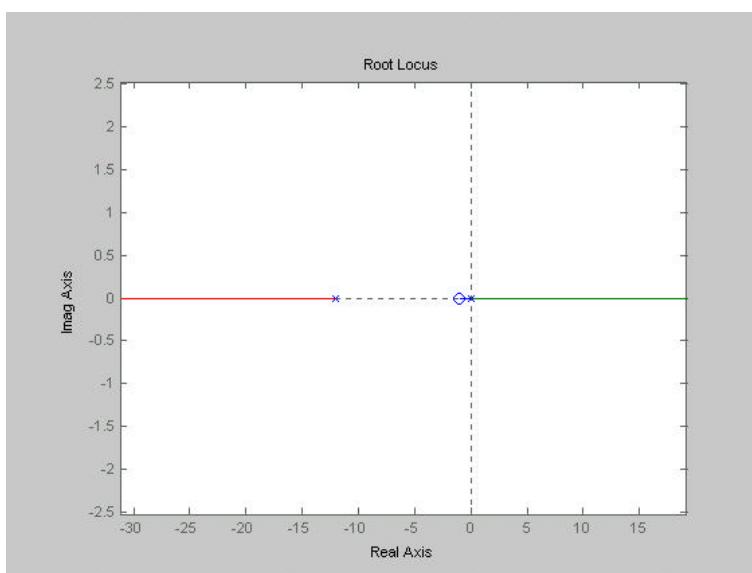
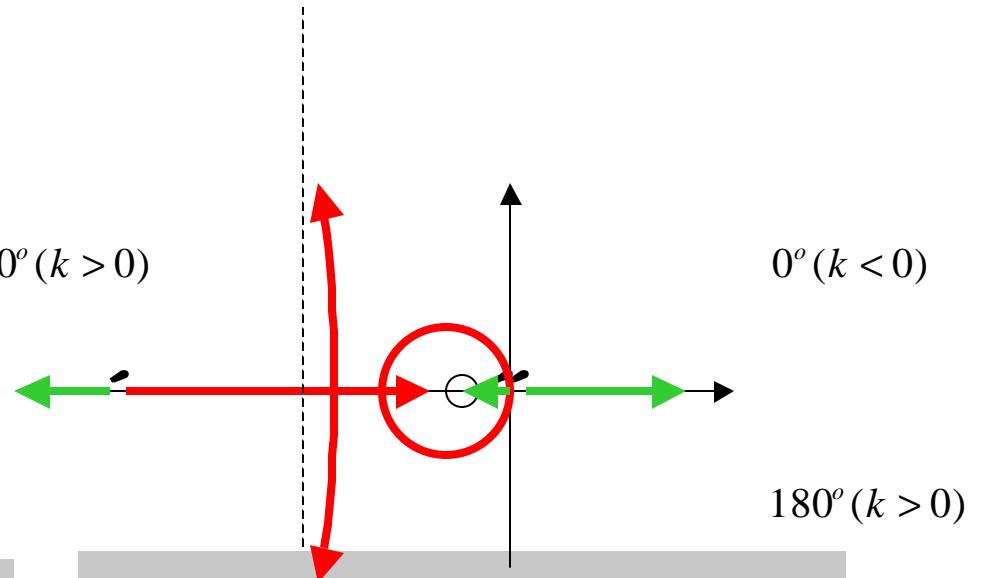
$$\alpha = -11/2$$

$$0^\circ(k < 0)$$

$$180^\circ(k > 0)$$

$$0^\circ(k < 0)$$

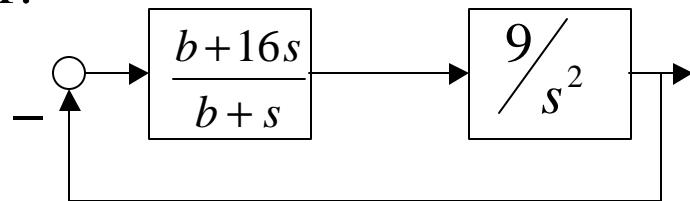
$$180^\circ(k > 0)$$



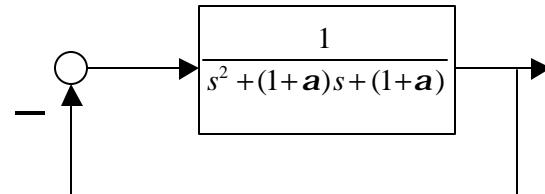
Today's Exercise

- sketch the complete root locus of the following systems first by hand and then compare with the MATLAB plot (use the rlocus command)

1.



3.



2.

