Today (10/23/01)

- Today
  - Gain/phase margin
  - lead/lag compensator
  - Ref. 6.4, 6.7, 6.10
- Reading Assignment: 6.3

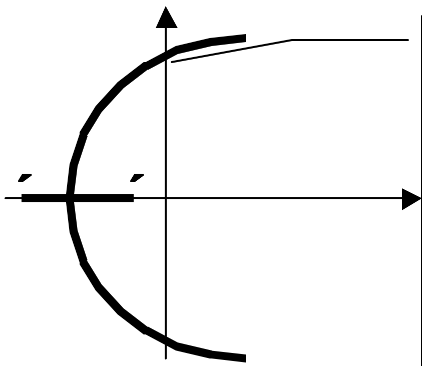
Last Time

In the last lecture, we discussed control design through shaping of the loop gain *GK*: keep *GK* large in the spectra of ref input & disturbance, keep *GK* small in the spectra of model uncertainty & sensor noise, keep *K* small for small control effort).

Today we will derive stability and robustness conditions in the frequency domain.

Stability of CL System

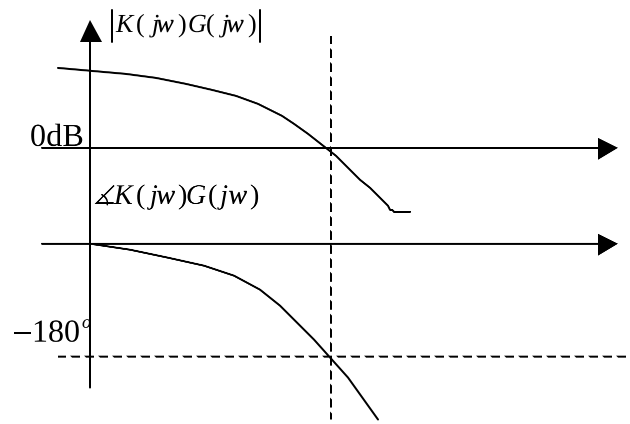
Consider an open loop stable system that becomes unstable with large gain:



At the point of instability,

$$\left|K(jw)G(jw)\right| = 1$$

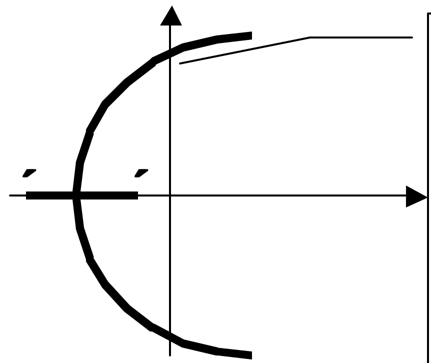
$$\measuredangle K(jw)G(jw) = \pm 180^{o}$$



Closed loop poles must satisfy:

$$(1 + K(s)G(s)) = 0 \Rightarrow \left|K(s)G(s)\right| = 1 \text{ and } \measuredangle K(s)G(s) = \pm 180^{o}$$

Stability Margins



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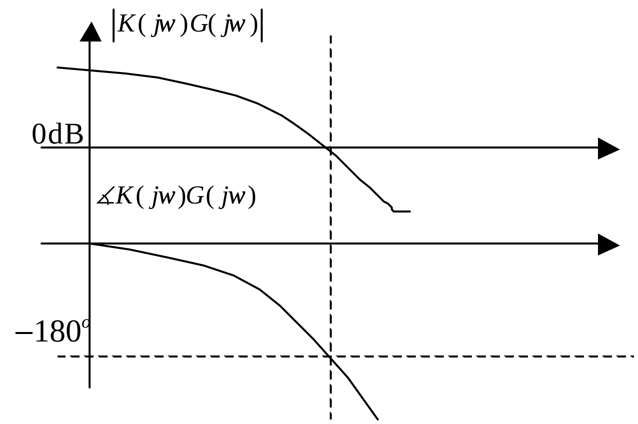
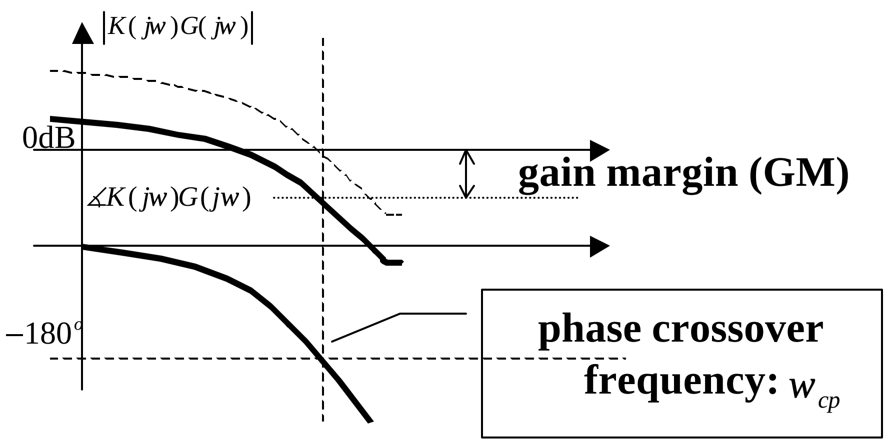
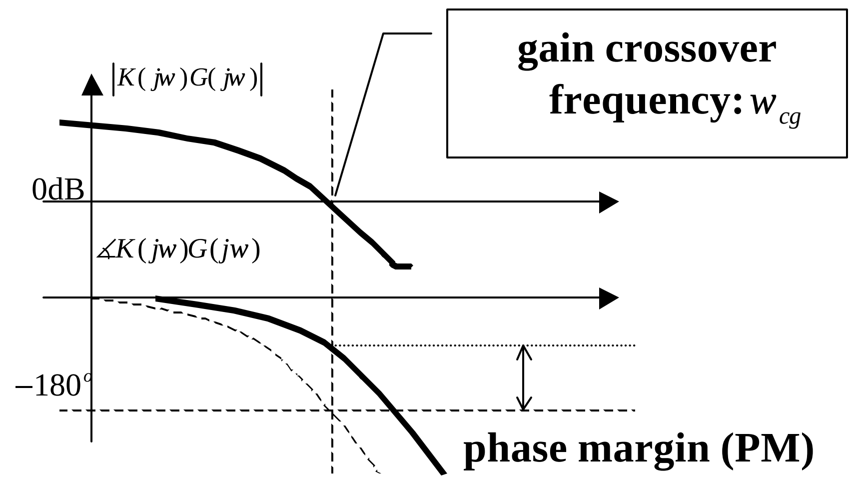


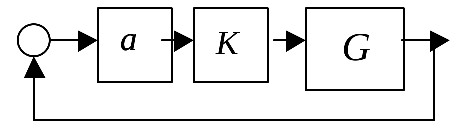
If actual gain is smaller:



If actual phase is closer to 0°:



Stability Margins



Gain margin means the amount of gain variation (a is a positive real number) that can be tolerated. If $a>1$, $(a)_{dB}>0$; if $a<1$, $(a)_{dB}<0$.

Phase margin means the amount of phase variation (a is a phase shift: $a=e^{jf}$) that can be tolerated.

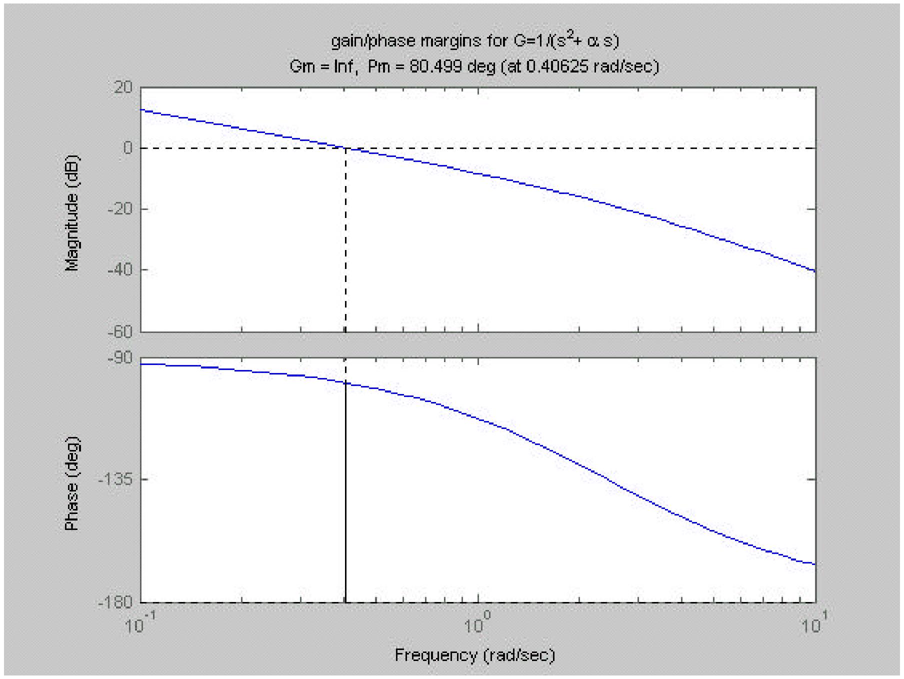
Stability Margins

What if there are multiple gain crossover and phase crossover points?

We will discuss Nyquist plot in the next lecture which will give us a definitive answer.

For now, use MATLAB `margin` command:

`margin(sys);`



Loop Shaping Perspective

If we can reduce the gain near the phase crossover, we can improve the gain margin (gain stabilization through gain roll-off).

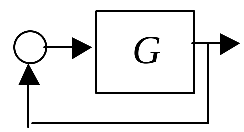
If we can increase the phase (add phase lead) near the gain cross over, we can improve the phase margin (phase stabilization through lead compensation).

Unfortunately, gain and phase are not independent. Hence, changing gain ➔ changing phase. We'll see this later in Bode gain/phase formula.

Relationship to Performance

How do we infer performance directly from the loop gain?

Based on the standard second order systems (with no zero), we have the following rules of thumb:



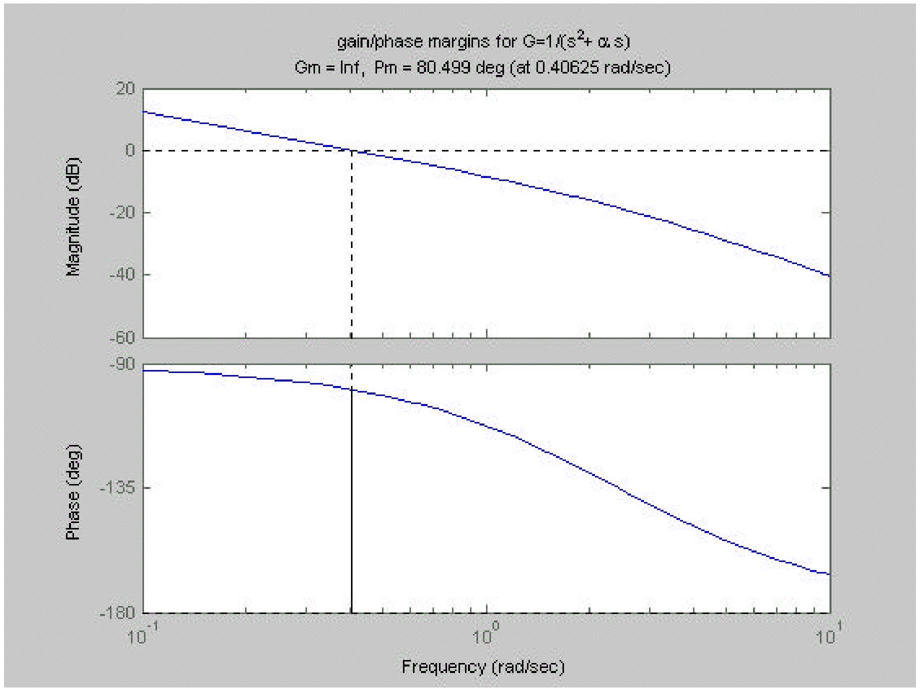
$$G(s) = \frac{w_n^2}{s^2 + 2Vw_n s}$$

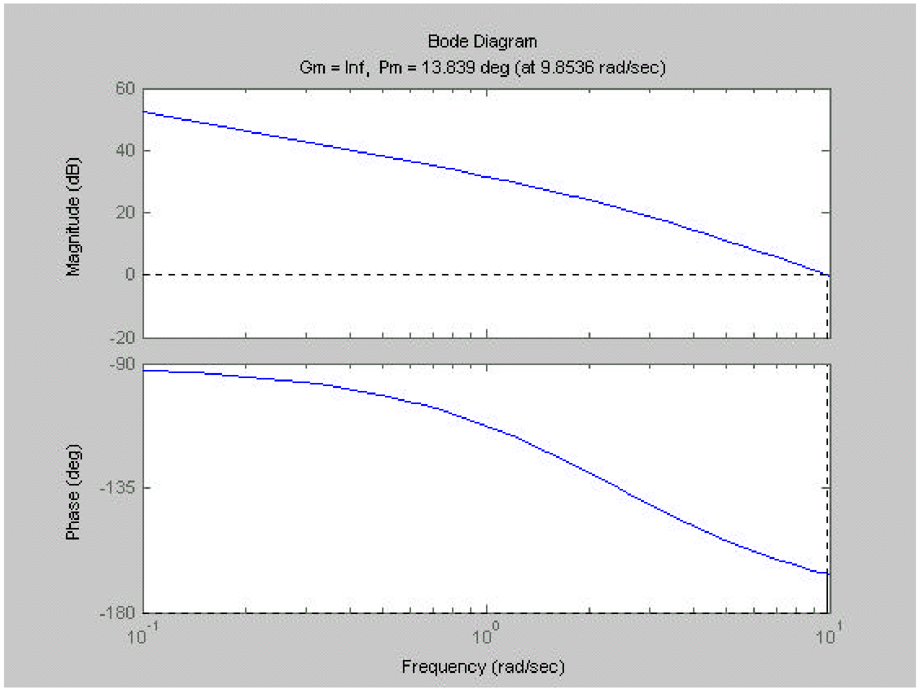
$$G_{cl}(s) = \frac{w_n^2}{s^2 + 2Vw_n s + w_n^2}$$

$$z \approx \frac{\text{PM}}{100} \quad \text{(e.g., } 30^{\circ} \text{ PM} \approx 30\% \text{ damping)}$$

$$w_{\text{n}} \approx w_{cg} \quad \text{(gain cross over frequency, i.e., BW)}$$

Example



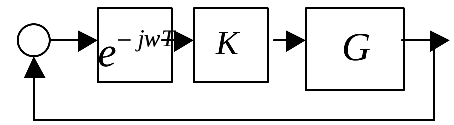


$k_p=1$, **PM=80.5°, $\mathbf{w_{cg}}$ = .41rad/sec**

$k_p=100$, **PM=13.8°, $\mathbf{w_{cg}}$ = 9.85rad/sec**

$$K(s)G(s) = \frac{k_p}{s^2 + as}$$

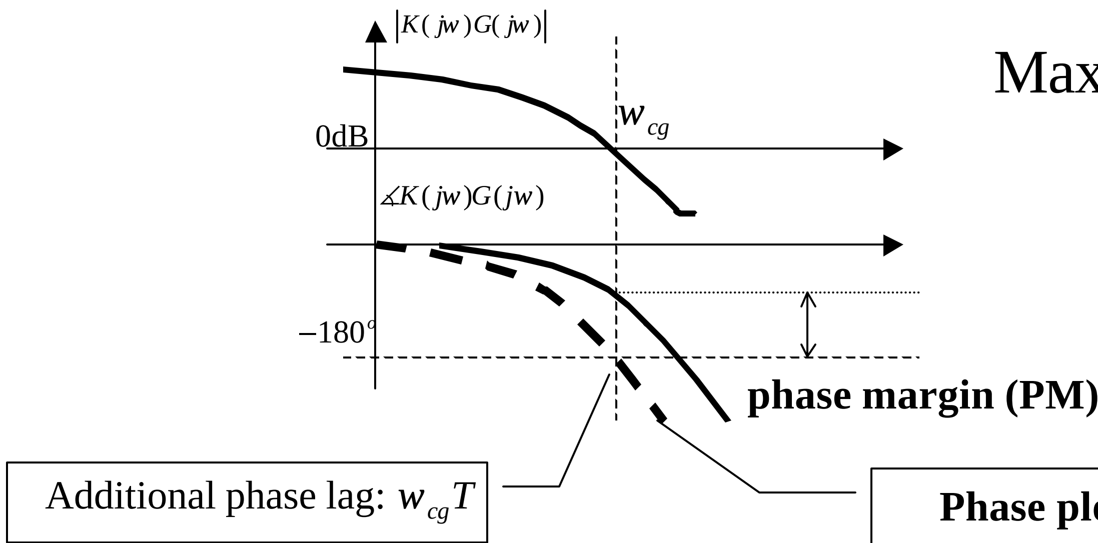
Time Delay



Time delay adds a phase shift of -*wT*.

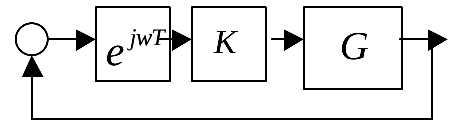
Boundary of stability: PM=$w_{cg}T$

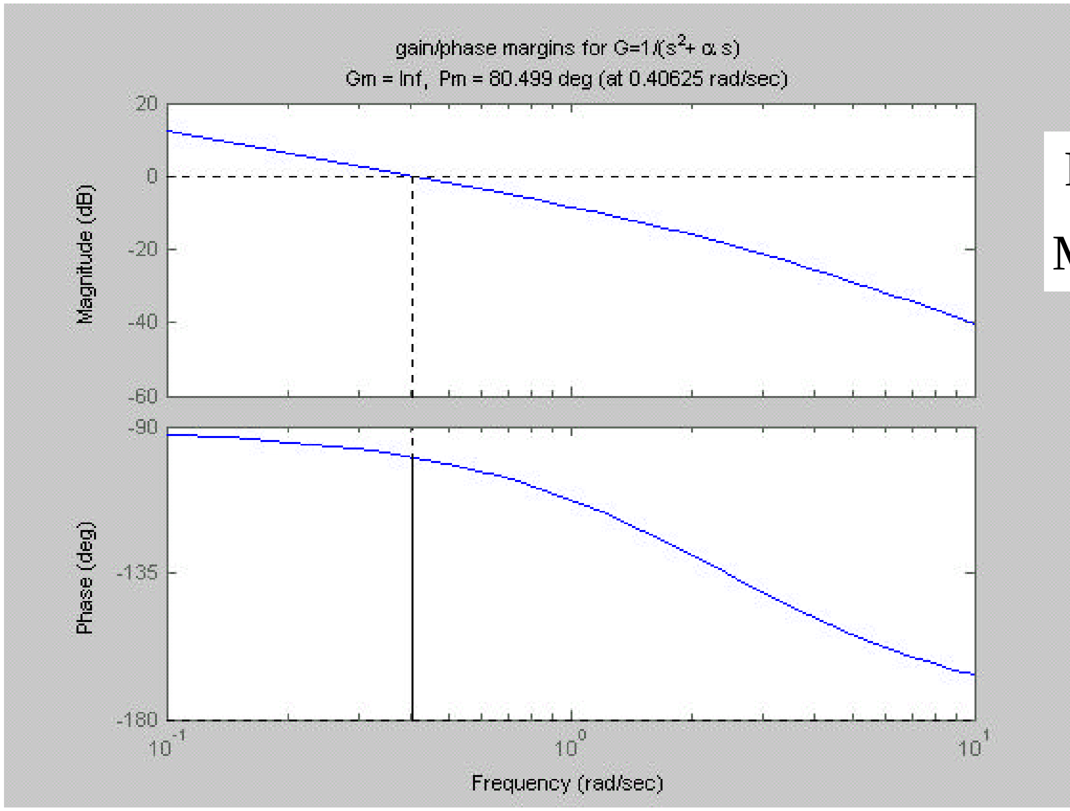
Maximum delay: $T_{\max} = \dfrac{\text{PM}}{w_{cg}}$



Phase plot with added phase lag from time delay.

Time Delay





$\text{PM}=80.5^{\circ}=1.405\text{rad}, w_{cg}=.41\text{rad/sec}$

Maximum delay: $T_{\max}=3.45\text{sec}$

```
G=tf(1,[1 Fv/Ic 0]);
T=3.46; % T=3.45
[n,d]=pade(3,T);Gd=tf(n,d);
max(real(pole(feedack(G*Gd,
1))))
```

```
T=3.46: 1.52e-5
T=3.45: -3.93e-4
```

Summary

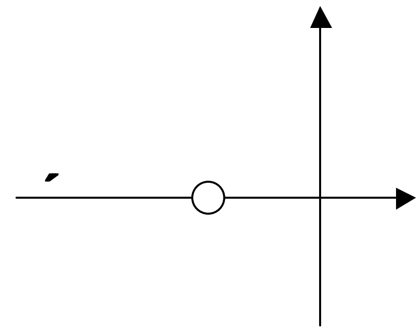
Frequency domain control design involves choosing $K(s)$ to achieve a loop gain $K(s)G(s)$ with the following attributes:

- large loop gain in spectra of r and d (e.g., for trajectory tracking and input disturbance rejection).
- small loop gain in spectra of n and Δ (e.g., for sensor noise rejection and unmodeled dynamics).
- small enough K to avoid actuator saturation
- adequate gain margin for gain robustness
- adequate phase margin for damping and time delay
- adequate bandwidth for speed of response

Lead Compensation

Lead filter can be used to add phase lead (improve phase margin) and increase bandwidth.

$$K_{lead}(s) = K\frac{Ts+1}{aTs+1}, a < 1$$



Three design parameters: K, a, T

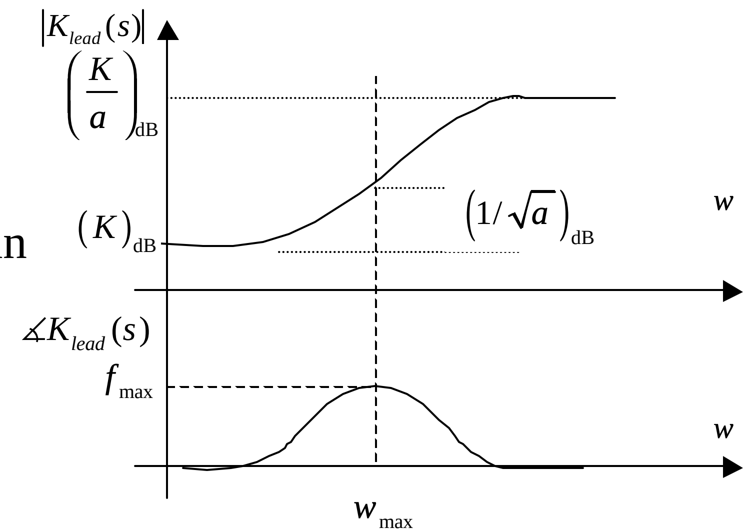
(i.e., overall gain, pole/zero locations)

key attributes:

max phase, location of max phase, high freq gain

$$a = \frac{1-\sin f_{\max}}{1+\sin f_{\max}}$$

$$w_{\max} = 1 \big/ T\sqrt{a}$$



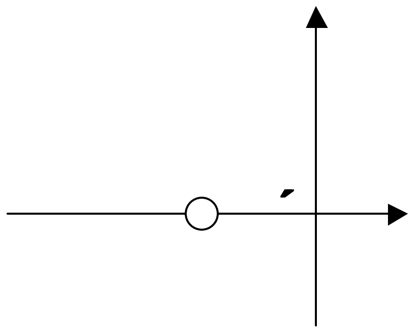
Lead Design Procedure

Given *G(s):*

- **Determine open loop gain *K* to meet low freq gain requirement (*KG(0)*) and/or bandwidth requirement (BW *KG(s)* about ½ of desired closed loop BW). Gain crossover freq=ω_{cg}.**
- **Evaluate PM of *KG(s)*. Determine extra phase lead needed, set it to ϕ_{max} .**
- **Determine α. Find the new gain crossover freq ω_{cg1}** $KG(j\omega_{cg1}) = \left(\sqrt{\alpha}\right)_{\mathrm{dB}}$
- **Let ω_{max}= ω_{cg1} and solve for *T.***
- **Check PM, BW of $G(s)K_{lead}(s)$ and iterate if necessary.**
- **Check all other specifications, and iterate design; add more lead compensators if necessary.**

Lag Compensation

Lag filter can be used to boost DC gain (to reduce steady state error; but can reduce phase margin.



$$K_{lag}(s) = a\,\frac{Ts+1}{aTs+1}, a > 1$$

Two design parameters: a, T

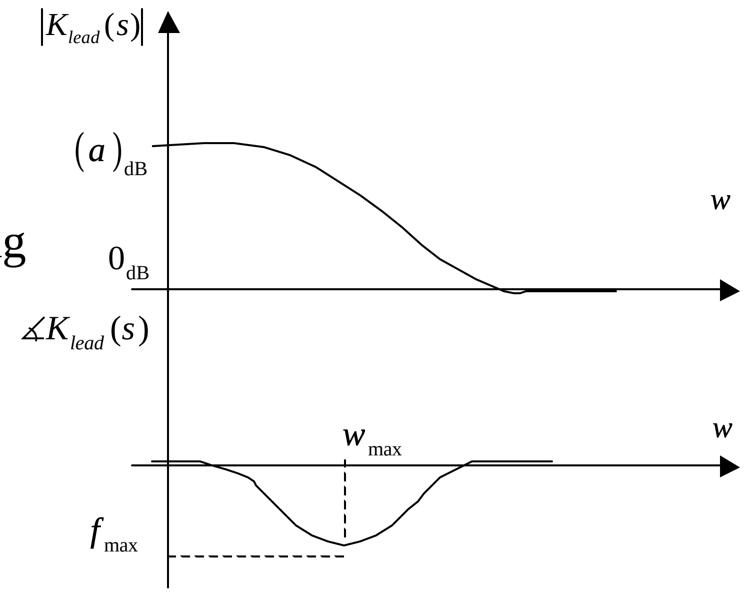
(i.e., pole/zero locations)

key attributes:

low freq gain, max phase lag, location of max lag

$$a = \frac{1\text{-}\sin f_{\max}}{1+\sin f_{\max}}, f_{\max} < 0$$

$$w_{\max} = \frac{1}{T\sqrt{a}}$$



Lag Design Procedure

Given *G(s)*:

- **Determine overall open loop gain *K* to meet PM requirement (without lag compensation).**
- **Determine a to achieve desired low frequency gain.**
- **Choose 1/*T* (zero location) to be 1 decade below gain crossover frequency of *KG(s)*.**
- **Check all other specifications, and iterate if necessary.**

Example

Example 6.15 in book:

Given $G(s)=1/(2s+1)(s+1)(.5*s+1)$, design a lead compensator so that DC gain = 9 and PM > 25^o

Exercise 10

Apply the lead filter design to G(s)=1/s/(s+Fv/Ic) as in Project 2 to achieve BW of at least 10rad/sec and phase margin of at least 60^o. Show the step responses of the this controller in both linear and nonlinear simulation.