Today (10/16/01)

- Today
  - Collect Project 2
  - Hand out Project 3
  - Frequency response
  - Ref. 6.1
- Reading Assignment: 6.9

Frequency Response

- **Consider a stable system driven by a sinusoidal input (like in Project 1).**

$$u(t) = A\sin(\omega t) \longrightarrow \boxed{G(s)} \xrightarrow{y(t)}$$

After the initial transient dies out,

$$y(t) = |G(j\omega_o)| U_o \sin(\omega_o t + \measuredangle G(j\omega_o))$$

$$\left.\begin{array}{l} \text{magnitude of } G\text{:} \quad |G(j\omega)| \\ \text{phase of } G\text{:} \ \measuredangle G(j\omega) \end{array}\right\} \text{Frequency response of } G$$

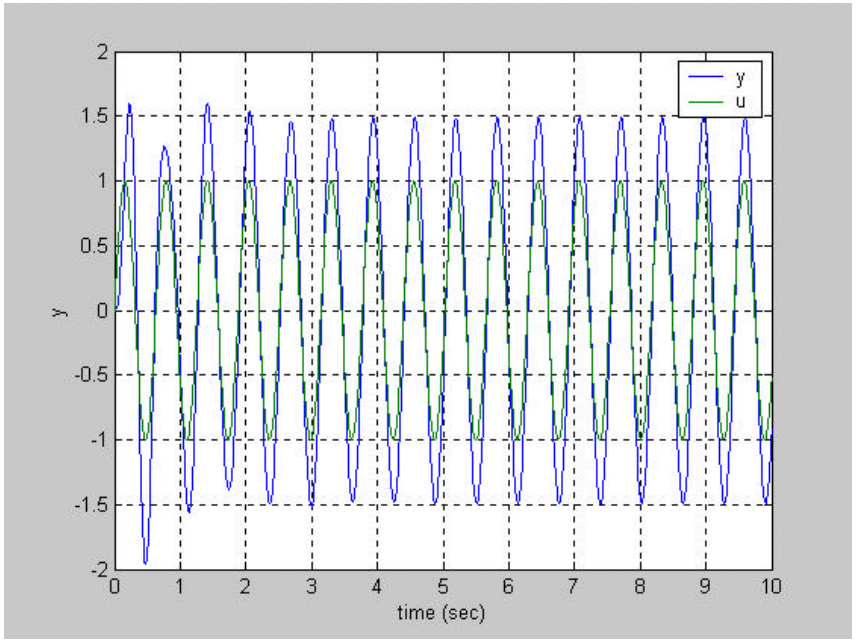
- **Let *u(jω)* be the Fourier transform of *u(t)* (spectrum of *u*). *G(jω)* can be considered as the gain scaling and phase shift of each sinusoidal component of *u.***

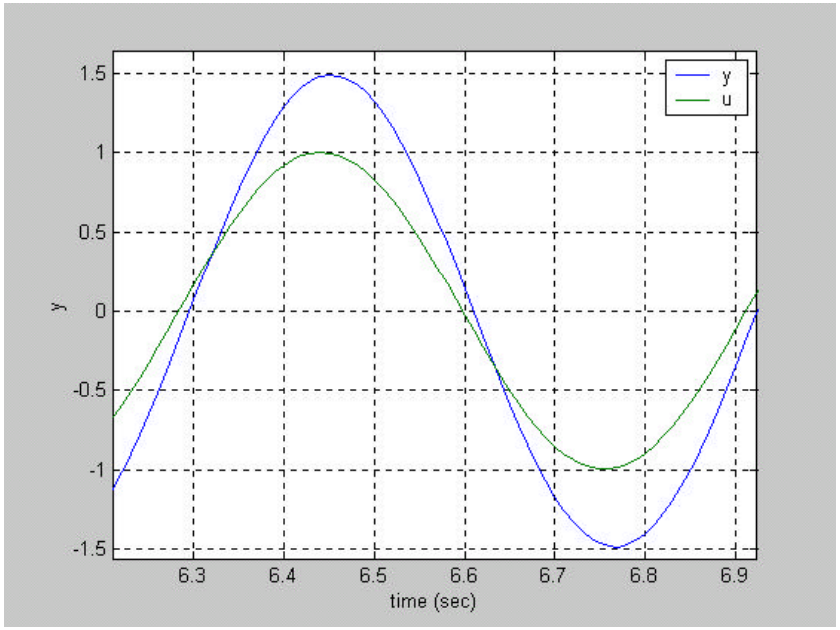
Example

$$u(t) = A\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$$

Let $G(s) = \dfrac{300}{s^2 + 2.427s + 300}$

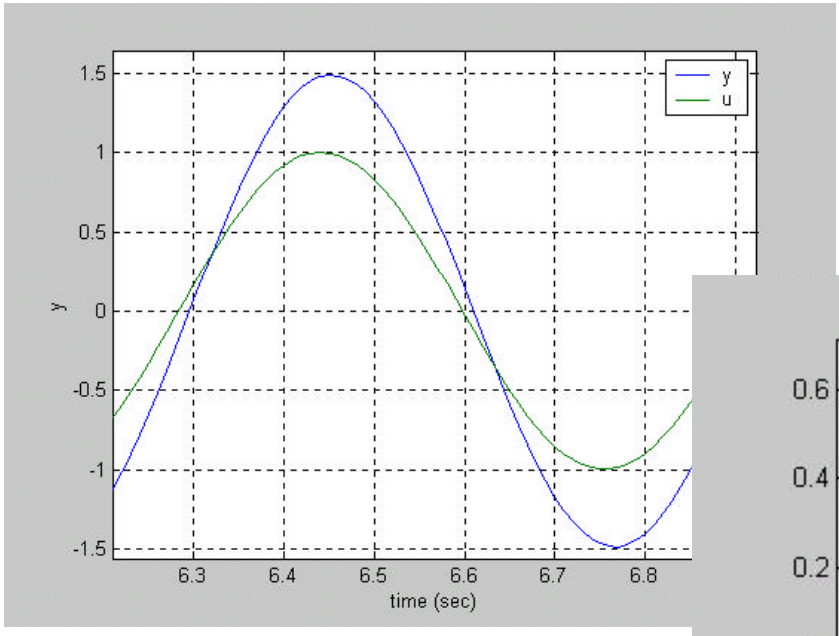
$$G(10j) = 1.49\exp(-j0.12)$$





Example

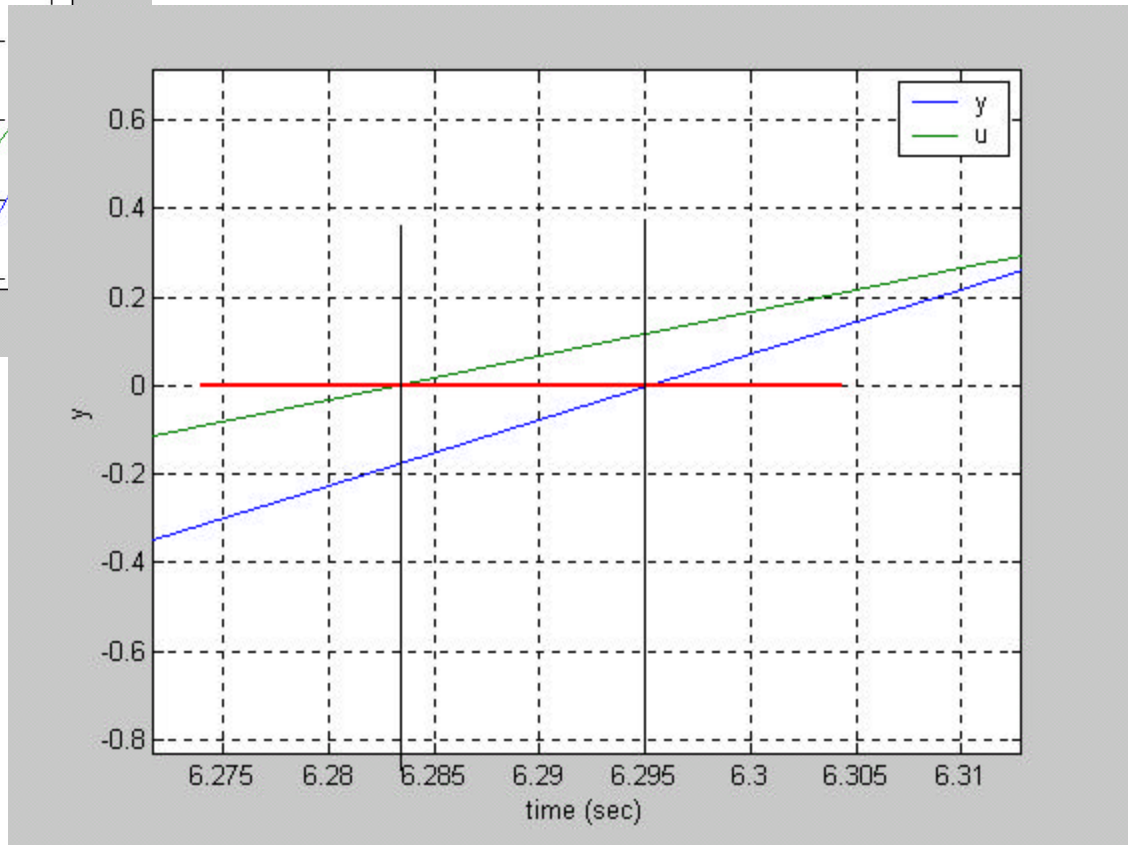
$$G(10j) = 1.49\exp(-j0.12)$$



$$\sin(10(t - \phi/10))$$

$\phi/10 =$ separation of zero crossings

$\phi/10 = 0.012$ rad or $\phi = 0.12$rad



peak of y / peak of $u \approx 1.5$

Example

```
a=2.4274;
G=tf(1,[1 a 0]);
G1=feedback(G*300,1);
G1_10=evalfr(G1,10*j);
disp(sprintf('gain of G(10j) = %g',abs(G1_10)));
disp(sprintf('phase of G(10j) = %g',angle(G1_10)));
t=(0:.01:10);
u=sin(10*t);
y=lsim(G1,u,t);
figure(1);plot(t,y,t,u);grid;
legend('y','u');xlabel('time (sec)');ylabel('y');
```

Bode Plot

- **Gain plot is represented in decibels (dB) of $|G(j\omega)|$ vs. $\log_{10}\omega$:** $|G|_{dB} = 20\log|G|$
- **Phase plot is angle($G(j\omega)$) vs. $\log_{10}\omega$ (in degree or radian)**

$$G_1G_2 = |G_1||G_2|\exp(j\measuredangle(G_1))\exp(j\measuredangle(G_2))$$

$$|G_1G_2| = |G_1||G_2| \quad \exp(j\measuredangle(G_1G_2)) = \exp(j(\measuredangle(G_1) + \measuredangle(G_2)))$$

- **Both gain/phase plots are additive**

$$|G_1G_2|_{dB} = 20\log|G_1G_2| = 20\log|G_1| + 20\log|G_1| = |G_1|_{dB} + |G_2|_{dB}$$

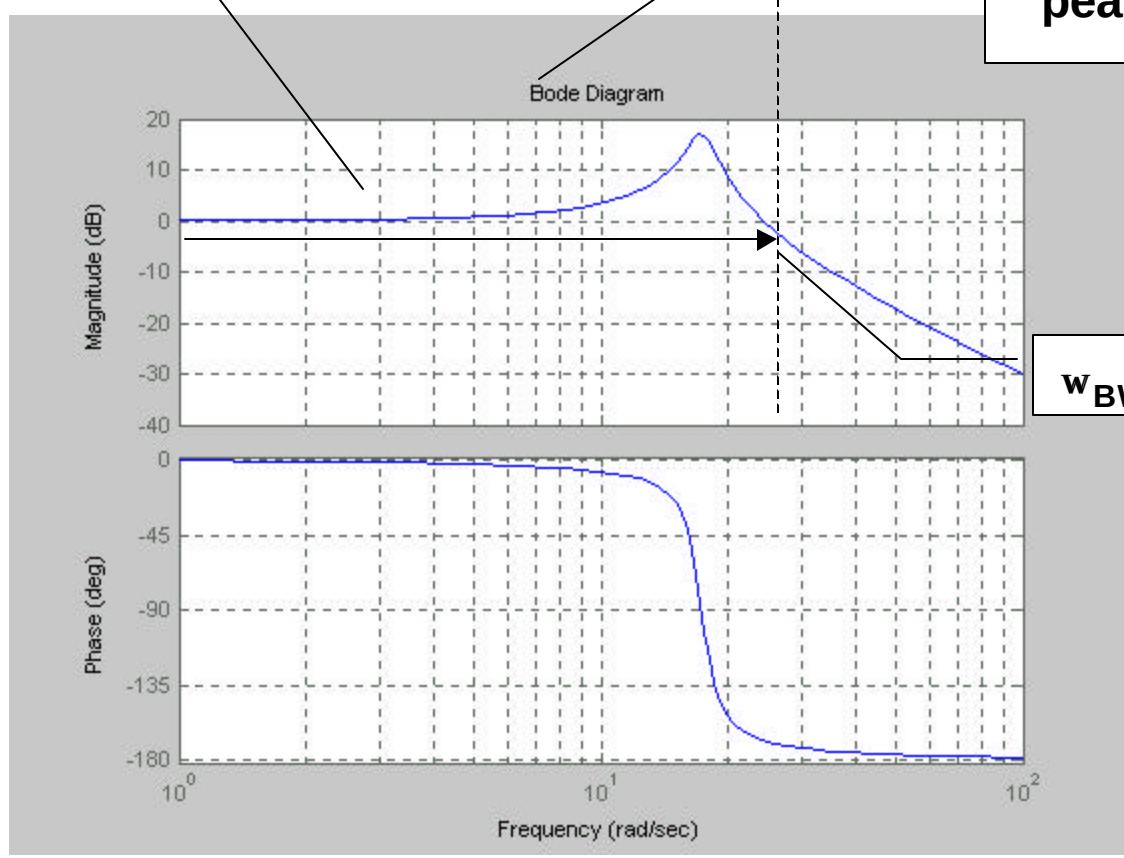
$$\measuredangle(G_1G_2) = \measuredangle(G_1) + \measuredangle(G_2)$$

Bode Plot

- **To visualize complete frequency spectrum of *G(s)*, use MATLAB command `bode(G)`**

bandwidth: frequency range over which input sinusoids can be satisfactorily tracked. Typically chosen as –3dB drop from DC value.

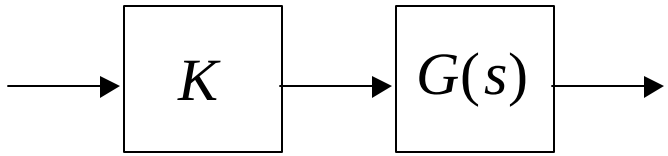
resonant peak M_r



w_{BW}

Bode Plot by hand

Consider Bode plot of loop gain $KG(s)$:



$$KG(s) = K\frac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots} = K_o\frac{\left(\frac{s}{z_1}-1\right)\left(\frac{s}{z_2}-1\right)\cdots}{\left(\frac{s}{p_1}-1\right)\left(\frac{s}{p_2}-1\right)\cdots}$$

$$\left|KG(j\omega)\right|_{dB} = \left|K_o\right|_{dB} + \left|j\omega/z_1 - 1\right|_{dB} + \left|j\omega/z_2 - 1\right|_{dB} + \cdots$$
$$- \left|j\omega/p_1 - 1\right|_{dB} - \left|j\omega/p_2 - 1\right|_{dB} - \cdots$$

$$\measuredangle\,(KG(j\omega)) = \measuredangle\,K_o + \measuredangle\,(j\omega/z_1 - 1) + \measuredangle\,(j\omega/z_2 - 1) + \cdots$$
$$- \measuredangle\,(j\omega/p_1 - 1) - \measuredangle\,(j\omega/p_2 - 1) - \cdots$$

Bode Plot by hand

Different types of terms:

0. constants: c

1. zeros/poles at origin: s^n

2. real zeros/poles: $(s\tau + 1)^{\pm 1}$

3. complex zeros/poles: $\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1\right]^{\pm 1}$

We will develop rules for each term. For the final Bode plot, just add the individual gain and phase plots together.

Bode Plot by Hand

$$|c|_{dB} = 20\log_{10}|c| = \text{constant}$$

$$\measuredangle\,(c) = \text{ constant phase} = \begin{cases} 0^o & \text{if } c > 0 \\ 180^o & \text{if } c < 0 \end{cases}$$

$$\left|(j\omega)^n\right|_{dB} = n \cdot 20\log_{10}|\omega| \;\; (20 \cdot n \text{ dB/decade})$$

$$\measuredangle\,(j\omega)^n = n \cdot 90^o \;\; (\text{constant phase})$$

Bode Plot by Hand

$j\omega\tau + 1$:

For $\omega\tau \ll 1$: $j\omega\tau + 1 \approx 1$ (0dB, 0 phase)

For $\omega\tau \gg 1$: $j\omega\tau + 1 \approx j\omega\tau$ (20dB/decade, 90°)

$\omega\tau = 1$: 3dB, 45°

Gain asymptotes: 0dB up to $\omega = 1/\tau$, 20dB/decade increase beyond

Phase asymptotes: Straight line approximation from 0° to 90° from 1 decade below $\omega = 1/\tau$ to 1 decade above $\omega = 1/\tau$, passing through 45° at $\omega = 1/\tau$.

Bode Plot by Hand

$$\beta(j\omega) = \left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1} :$$

For $\omega \ll \omega_n : \beta(j\omega) \approx 1$ (0dB, 0 phase)

For $\omega \gg \omega_n : \beta(j\omega) \approx \left[-\left(\frac{\omega}{\omega_n}\right)^2\right]^{\pm 1}$ (40dB/decade, $\pm 180^o$)

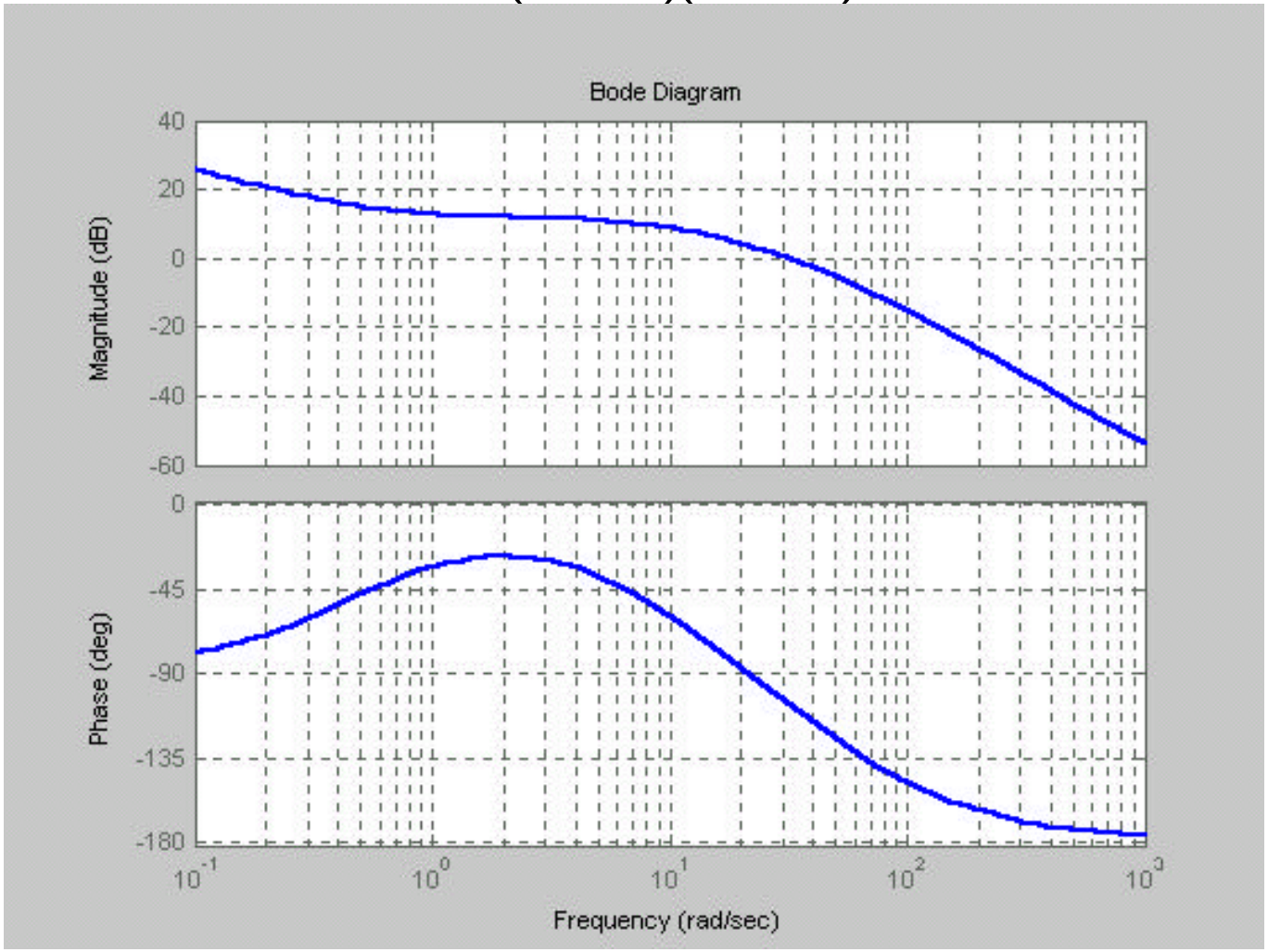
$\omega = \omega_n$: $\pm|2\zeta|$ (small $\zeta \Rightarrow$ large valley/peak) , 90°

Gain asymptotes: 0dB up to $\omega = \omega_n$, 40dB/decade inc/dec beyond

Phase asymptotes: Straight line approximation from 0° to $\pm 180^o$ from 1 decade below $\omega = \omega_n$ to 1 decade above $\omega = \omega_n$, passing through 90° at $\omega = \omega_n$.

Example

$$KG(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$



Exercise 8

Sketch by hand the Bode plot of
$G(s)=1/(s^2+as)$

Sketch by hand the Bode plot of lead filter: $G(s)=K(Ts+1)/(\alpha Ts+1) \quad \alpha<1$

Sketch by hand the Bode plot of lag filter:
$G(s)=\alpha(Ts+1)/(\alpha Ts+1) \quad \alpha>1$